

The orthogonal subcategory problem in homotopy theory

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ABSTRACT. The statement that localization with respect to any (possibly proper) class of morphisms exists in any locally presentable category cannot be proved using the ordinary ZFC axioms of set theory. In fact, it is known to be equivalent to a large-cardinal principle. In this article we show that, if Vopěnka's principle is assumed true, then in any cofibrantly generated, left proper, simplicial model category \mathcal{M} whose underlying category is locally presentable, homotopy localization exists with respect to any class of maps. We also show that, in any such category, every homotopy idempotent functor is a homotopy localization with respect to some class \mathcal{S} of maps. Furthermore, if Vopěnka's principle holds, then \mathcal{S} can be chosen to be a set. There are examples showing that the latter need not be true if \mathcal{M} is not cofibrantly generated. The above assumptions on \mathcal{M} are satisfied by simplicial sets and symmetric spectra over simplicial sets, among many other model categories.

Introduction

Locally presentable categories were introduced by Gabriel and Ulmer in [18]. This concept has proved to be very useful in category theory. Among other things, the orthogonal subcategory problem (asking if localization with respect to every class of morphisms exists) has a positive solution in locally presentable categories if the given class of morphisms is a set; see, e.g., [1, 1.37]. Moreover, if one assumes the validity of a suitable set-theoretical principle, then there is also a positive solution for any proper class of maps. In fact, Adámek, Rosický and Trnková proved in [2] that a positive answer to the orthogonal subcategory problem in locally presentable categories is equivalent to the weak Vopěnka principle, a large-cardinal principle that cannot be proved using the usual ZFC axioms (Zermelo–Fraenkel axioms with the axiom of choice).

Localizing with respect to sets of maps is a common technique in homotopy theory, as well as in other areas of Mathematics. However, localizing with respect to proper classes of maps is a more delicate issue, since the standard methods may fall into set-theoretical difficulties (see for instance [9], where positive results in equivariant homotopy theory involving localization with respect to proper classes

2000 *Mathematics Subject Classification.* Primary 55U35; Secondary 55P60, 18G55.

Key words and phrases. Model category, localization, large cardinal principle.

During the preparation of this paper, the second-named author was a fellow of the Marie Curie Training Site with reference no. HPMT-CT-2000-00075 of the European Commission, hosted by the CRM, Barcelona. The first-named author was supported by the Spanish Ministry of Education and Science under MEC-FEDER grant MTM2004-03629.

of maps were obtained). Due to difficulties of this sort, it is still unknown whether the existence of arbitrary cohomological localizations of spaces can be proved or not using the ZFC axioms. An interesting step was made in [8], based on results in [1], by showing that Vopěnka's principle implies the existence of localization with respect to any proper class of maps in the category of simplicial sets. The existence of cohomological localizations is of course a special case. Vopěnka's principle is equivalent to the statement that the category of ordinals cannot be fully embedded into the category of graphs (where a graph is meant to be a binary relation). This statement has a place in the hierarchy of large-cardinal principles; see [1].

In this article we contribute further to the ongoing program of extending basic results from locally presentable categories to homotopy theory, which may perhaps give answers to other open problems, under large-cardinal assumptions. In order to achieve this, one has to work in suitable model categories. Specifically, our results are stated in left proper, combinatorial, simplicial model categories. The term "combinatorial" means that the model category is cofibrantly generated and the underlying category is locally presentable (see [1] and [20] for the definitions of these concepts). This notion is due to J. H. Smith, who constructed (in unpublished work) localizations of combinatorial model category structures with respect to sets of maps. In [15], Dugger proved that every combinatorial model category is equivalent to a localization of a category of diagrams of simplicial sets, hence generalizing [1, 1.46]. Among many other examples, the model category of simplicial sets and the model category of symmetric spectra based on simplicial sets are combinatorial.

In Section 1 we show that Vopěnka's principle implies the existence of localization with respect to any class of maps in left proper, combinatorial, simplicial model categories. This fact can also be deduced, with a different argument, from results obtained by Rosický and Tholen in [24, §2]. Furthermore, under Vopěnka's principle, any such localization is equivalent to localization with respect to some set of maps.

Next, we address a closely related question, raised by Dror Farjoun in [11], asking if any functor L on simplicial sets that is idempotent up to homotopy is equivalent to localization with respect to some map f . He himself showed in [12] that, if L is assumed to be, in addition, *continuous*, then it is indeed equivalent to localization with respect to a *proper class* of maps. This result was improved in [8] by showing that the assumption that L be continuous is unnecessary, and that, under Vopěnka's principle, the proper class of maps defining L can be replaced by a set. Furthermore, it was shown that such a replacement of a class by a set cannot be done in general using only the ZFC axioms, since a counterexample was exhibited by means of another assumption (the nonexistence of measurable cardinals), which is relatively consistent with ZFC.

In Section 2 we show (without resorting to large-cardinal principles) that every homotopy idempotent functor L in a simplicial model category \mathcal{M} is equivalent to localization with respect to a proper class of maps, assuming either that L is continuous or that \mathcal{M} satisfies suitable hypotheses allowing to approximate any homotopy functor by a continuous functor. For this, one may assume that \mathcal{M} is a simplicial model category that is proper, cofibrantly generated and stable (as in [23]), or left proper and either combinatorial or cellular (as in [14]). Furthermore, if one assumes that Vopěnka's principle is true and \mathcal{M} is combinatorial, then, again, the proper class of maps defining L can be replaced by a set. In most cases of

interest, such a set of maps can further be replaced by a single map (by taking the coproduct of all maps in the set), but not always, as we show by means of an example at the end of the paper.

In [10] an example was given of a homotopy idempotent functor in a locally presentable (but not cofibrantly generated) model category that fails to be a localization with respect to any set of maps. Namely, in the category of maps between simplicial sets with the model structure generated by the collection of orbits (as defined in [13]), the functor that sends every map to the final object (i.e., a map between two points) is not a localization with respect to any set of maps. Hence our results in Section 2 below are sharp.

Acknowledgements: Discussions with Mark Hovey, Jiří Rosický, Brooke Shipley and Jeff Smith are greatly appreciated.

1. Simplicial orthogonality

Model categories were introduced by Quillen in [22] and have recently been discussed in the books [16], [19], [20], [21], among many other places, with slight changes in the terminology and even in the assumptions. In this article we will assume that model categories are complete, cocomplete, and equipped with functorial factorizations. See [16, § 9], [20, § 7], or [21, § 1] for more details.

Although our main results are stated for *simplicial* model categories (for the definition, see for example [19, II.3] or [20, 9.1.5]), several of our steps require only the use of *homotopy function complexes*, as introduced in [17] and discussed in [20, Ch. 17] or [21, § 5]. Thus, for any given model category \mathcal{M} , we make a functorial choice of a fibrant simplicial set $\text{map}(X, Y)$ for each X and Y in \mathcal{M} , whose homotopy type is the same as the diagonal of the bisimplicial set $\mathcal{M}(X^*, Y_*)$ where $X^* \rightarrow X$ is a cosimplicial resolution of X and $Y \rightarrow Y_*$ is a simplicial resolution of Y . The homotopy type of $\text{map}(X, Y)$ remains unchanged if X or Y are replaced by weakly equivalent objects. If \mathcal{M} is a simplicial model category and $\text{Map}(X, Y)$ denotes the simplicial set given as part of the structure in \mathcal{M} , then $\text{Map}(QX, RY)$ is a good choice of a homotopy function complex, where Q is a cofibrant approximation functor and R is a fibrant approximation functor in \mathcal{M} .

Before discussing simplicial orthogonality in model categories by means of homotopy function complexes, we recall the following older concepts from category theory. If \mathcal{C} is any category, an object X and a morphism $f: A \rightarrow B$ are called *orthogonal* (see [1] or [7] for details and motivation) if the induced function

$$f^*: \mathcal{C}(B, X) \longrightarrow \mathcal{C}(A, X)$$

is bijective. (We denote by $\mathcal{C}(X, Y)$ the set of morphisms from X to Y in \mathcal{C} .) If L is an endofunctor of \mathcal{C} equipped with a natural transformation $\eta: \text{Id} \rightarrow L$ such that $L\eta: L \rightarrow LL$ is an isomorphism and $\eta L = L\eta$, then L is called an *idempotent functor* or a *localization*. Then every object isomorphic to LX for some X is orthogonal to every morphism f such that Lf is an isomorphism, and these two classes determine each other by the orthogonality relation; that is, an object is isomorphic to LX for some X if and only if it is orthogonal to all morphisms f such that Lf is an isomorphism, and reciprocally.

As a special case, this terminology applies to the homotopy category $\text{Ho}\mathcal{M}$ associated with any model category \mathcal{M} . Thus, orthogonality in $\text{Ho}\mathcal{M}$ between an

object X and a map $f: A \rightarrow B$ amounts to the condition that

$$(1) \quad f^*: [B, X] \longrightarrow [A, X]$$

be bijective, where $[X, Y]$ means, as usual, $\text{HoM}(X, Y)$. Examples of idempotent functors in the homotopy category of simplicial sets, such as homological localizations, have been studied since several decades ago; see [4].

Throughout the extensive study of localizations undertaken since then in homotopy theory, a stronger notion of “simplicially enriched orthogonality” came to be considered. There is no widely agreed terminology for it yet. It was called *simplicial orthogonality* in [8] and *homotopy orthogonality* in [20, §17]. Thus, if \mathcal{M} is any model category with a choice of homotopy function complexes, an object X and a map $f: A \rightarrow B$ will be called *simplicially orthogonal* or *homotopy orthogonal* (not to be confused with orthogonality in HoM) if the induced map of simplicial sets

$$(2) \quad f^*: \text{map}(B, X) \longrightarrow \text{map}(A, X)$$

is a weak equivalence. Since there is a natural bijection between $\pi_0 \text{map}(X, Y)$ and $[X, Y]$, homotopy orthogonality implies indeed orthogonality in HoM . Although plenty of examples show that the converse is not true, we discuss in Section 2 an important situation where the converse holds.

The fibrant objects that are homotopy orthogonal to a given map f are usually called *f-local*. More generally, if \mathcal{S} is any class of maps, the fibrant objects that are homotopy orthogonal to all the maps in \mathcal{S} are called *\mathcal{S} -local*. We denote by \mathcal{S}^{\perp} the closure under weak equivalences of the class of \mathcal{S} -local objects, and call it the *homotopy orthogonal complement* of \mathcal{S} . Similarly, for a class \mathcal{D} of objects, we denote by \mathcal{D}^{\perp} the class of maps that are homotopy orthogonal to all the objects in \mathcal{D} . In particular, the maps in $(\mathcal{S}^{\perp})^{\perp}$ are called *\mathcal{S} -local equivalences*, or shortly *\mathcal{S} -equivalences*.

A *homotopy localization* is an endofunctor $L: \mathcal{M} \rightarrow \mathcal{M}$ preserving weak equivalences, taking fibrant values, and equipped with a natural transformation $\eta: \text{Id} \rightarrow L$ (called a *coaugmentation*) which is idempotent up to homotopy; that is, for each object X , the morphisms $L\eta_X$ and η_{LX} from LX to LLX coincide in HoM and are weak equivalences. Thus L defines indeed a localization in HoM .

If L is a homotopy localization such that LX is \mathcal{S} -local and $\eta_X: X \rightarrow LX$ is an \mathcal{S} -equivalence for all X , where \mathcal{S} is any class of maps, then we say that L is a homotopy localization with respect to the class \mathcal{S} , or shortly an *\mathcal{S} -localization*.

The *orthogonal subcategory problem* in homotopy theory asks if an \mathcal{S} -localization exists for every class \mathcal{S} of maps in a model category \mathcal{M} . One reason for using (2) instead of (1) as orthogonality relation is the fact that the answer to the orthogonal subcategory problem would too often be negative using (1). For instance, there is no localization in the homotopy category of simplicial sets onto the class of simply connected spaces. See [5] for a more elaborate counterexample.

It is well known that the orthogonal subcategory problem has a positive solution whenever \mathcal{S} is a set and \mathcal{M} satisfies certain assumptions, which vary slightly depending on the authors. We will call a model category *combinatorial* if it is cofibrantly generated and the underlying category is locally presentable. The definition of a locally presentable category can be found in [1] or [18], and the definition of a cofibrantly generated model category is contained, e.g., in [20]. The notion of properness is also discussed in [20].

THEOREM 1.1. *Let \mathcal{M} be a left proper, combinatorial, simplicial model category. For any set of maps \mathcal{S} there is a homotopy localization with respect to \mathcal{S} .*

PROOF. The core of the proof is in [3]. See [20] for an updated approach. \square

As far as we know, there is no way to prove this when \mathcal{S} is a proper class, not even for simplicial sets, using the ordinary axioms of set theory. In [8] it was shown that the statement of Theorem 1.1 holds for a proper class \mathcal{S} in the category of simplicial sets using a suitable large-cardinal axiom (Vopěnka's principle). We now undertake a generalization of this fact to other model categories.

LEMMA 1.2. *Given a cofibrantly generated simplicial model category \mathcal{M} and a small category \mathcal{C} , consider the projective simplicial model structure on the category of functors $\mathcal{M}^{\mathcal{C}}$ described in [20, Theorem 11.7.3]. Suppose that A is a cofibrant diagram in this model structure and X is a fibrant object of \mathcal{M} , then the \mathcal{C}^{op} -diagram $\text{hom}(A, X)$ of simplicial sets is fibrant in the injective model structure on $\mathcal{S}^{\mathcal{C}^{\text{op}}}$.*

PROOF. We have to show that any commutative square

$$\begin{array}{ccc} C & \longrightarrow & \text{hom}(A, X) \\ \downarrow i & & \downarrow \\ D & \longrightarrow & *, \end{array}$$

where i is an injective (objectwise) trivial cofibration of \mathcal{C}^{op} -diagrams of simplicial sets, admits a lift. By adjunction this problem is equivalent to finding a lift in the following commutative square in \mathcal{M} :

$$\begin{array}{ccc} A \otimes_{\mathcal{C}} C & \longrightarrow & X \\ \downarrow & & \downarrow \\ A \otimes_{\mathcal{C}} D & \longrightarrow & *. \end{array}$$

And this problem is equivalent, by another adjunction, to finding a lift in the following commutative square in $\mathcal{M}^{\mathcal{C}}$:

$$\begin{array}{ccc} \emptyset & \longrightarrow & X^D \\ \downarrow & & \downarrow i^* \\ A & \longrightarrow & X^C. \end{array}$$

In the last square the lift exists, since A is projectively cofibrant and i^* is an objectwise trivial, i.e. projective, fibration. \square

Recall that a partially ordered set A is called λ -directed, where λ is a regular cardinal, if every subset of A of cardinality smaller than λ has an upper bound.

LEMMA 1.3. *Let \mathcal{D} be any class of objects in a combinatorial simplicial model category \mathcal{M} , and let \mathcal{S} be its homotopy orthogonal complement. Then there exists a regular cardinal λ such that \mathcal{S} is closed under λ -directed colimits in the category of maps of \mathcal{M} .*

PROOF. Let I be a set of generating cofibrations for the model category \mathcal{M} . Choose a regular cardinal λ such that any object of the set of domains and codomains of maps in I is λ -presentable (such a cardinal exists since the category \mathcal{M} is locally

presentable). Let A be any λ -directed partially ordered set, and suppose given a diagram $f: A \rightarrow \text{Arr}\mathcal{M}$, where $\text{Arr}\mathcal{M}$ is the category of maps in \mathcal{M} . Let us depict it, for simplicity, as a chain:

$$(3) \quad \begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & \cdots \\ f_0 \downarrow & & f_1 \downarrow & & & & f_n \downarrow & & \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & \cdots & \longrightarrow & Y_n & \longrightarrow & \cdots \end{array}$$

Suppose that the maps f_i are in \mathcal{S} for each $i \in A$. Since \mathcal{M} is cocomplete, $\text{Arr}\mathcal{M}$ is cocomplete as well, and we may consider the colimit of the diagram f . We need to show that the induced map $\text{colim } f_i: \text{colim } X_i \rightarrow \text{colim } Y_i$ is also in \mathcal{S} .

Consider the category \mathcal{M}^A of A -indexed diagrams in \mathcal{M} , and endow it with a model structure as described in [20, 11.6]. Thus, weak equivalences and fibrations are objectwise, and cofibrations are retracts of free cell complexes. The diagram (3) may be viewed as a single map in \mathcal{M}^A . Apply the cofibrant approximation functor to this map using the above model structure, hence obtaining the following commutative diagram in \mathcal{M} :

$$\begin{array}{ccccccc} \tilde{X}_0 & \xrightarrow{\hookrightarrow} & \tilde{X}_1 & \xrightarrow{\hookrightarrow} & \cdots & \xrightarrow{\hookrightarrow} & \tilde{X}_n & \xrightarrow{\hookrightarrow} & \cdots \\ \downarrow \tilde{f}_0 & \searrow & \downarrow \tilde{f}_1 & \searrow & & & \downarrow \tilde{f}_n & \searrow & \\ X_0 & \xrightarrow{\quad} & X_1 & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & X_n & \xrightarrow{\quad} & \cdots \\ \downarrow f_0 & \searrow & \downarrow f_1 & \searrow & & & \downarrow f_n & \searrow & \\ \tilde{Y}_0 & \xrightarrow{\hookrightarrow} & \tilde{Y}_1 & \xrightarrow{\hookrightarrow} & \cdots & \xrightarrow{\hookrightarrow} & \tilde{Y}_n & \xrightarrow{\hookrightarrow} & \cdots \\ \downarrow f_0 & \searrow & \downarrow f_1 & \searrow & & & \downarrow f_n & \searrow & \\ Y_0 & \xrightarrow{\quad} & Y_1 & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & Y_n & \xrightarrow{\quad} & \cdots \end{array}$$

where \tilde{f}_i is a cofibrant approximation of f_i .

For every $Z \in \mathcal{D}$, let \hat{Z} be a fibrant approximation to Z . The induced map

$$\text{Map}(\text{colim } \tilde{f}_i, \hat{Z}): \text{Map}(\text{colim } \tilde{Y}_i, \hat{Z}) \rightarrow \text{Map}(\text{colim } \tilde{X}_i, \hat{Z})$$

can be written as a limit of a fibrant diagram of maps of simplicial sets

$$\lim \text{Map}(\tilde{f}_i, \hat{Z}): \lim \text{Map}(\tilde{Y}_i, \hat{Z}) \rightarrow \lim \text{Map}(\tilde{X}_i, \hat{Z}).$$

The A^{op} -diagrams of simplicial sets $\text{Map}(\tilde{X}, \hat{Z})$ and $\text{Map}(\tilde{Y}, \hat{Z})$ are fibrant in the injective model structure by Lemma 1.2, therefore their inverse limits are homotopy inverse limits (the constant diagram of points is cofibrant in the injective model structure). Hence, $\text{Map}(\text{colim } \tilde{f}_i, \hat{Z}) = \lim \text{Map}(\tilde{f}_i, \hat{Z})$ is a weak equivalence, as a map induced between homotopy inverse limits by levelwise weak equivalences $\text{Map}(\tilde{f}_i, \hat{Z})$. This shows that $\text{colim } \tilde{f}_i$ is in \mathcal{S} .

Trivial fibrations in \mathcal{M} are preserved under λ -directed colimits, since the set of generating cofibrations has λ -presentable domains and codomains. From the commutative diagram

$$\begin{array}{ccc} \text{colim } \tilde{X}_i & \xrightarrow{\sim} & \text{colim } X_i \\ \downarrow \text{colim } \tilde{f}_i & & \downarrow \text{colim } f_i \\ \text{colim } \tilde{Y}_i & \xrightarrow{\sim} & \text{colim } Y_i \end{array}$$

we conclude that the map $\text{colim } \tilde{f}_i$ is a cofibrant approximation of the map $\text{colim } f_i$, since both $\text{colim } \tilde{X}_i$ and $\text{colim } \tilde{Y}_i$ are cofibrant in \mathcal{M} (\tilde{X} and \tilde{Y} are cofibrant diagrams

in \mathcal{M}^A and the colimit functor $\mathcal{M}^A \rightarrow \mathcal{M}$ is left Quillen by [20, 11.6.8(1)]. Hence, $\text{colim } f_i$ is in \mathcal{S} , as claimed. \square

The statement of Vopěnka's principle and enough motivation for its use in this context can be found in [1], [2], [8], and [24].

LEMMA 1.4. *Suppose that Vopěnka's principle is true. Let \mathcal{D} be any class of objects in a combinatorial simplicial model category \mathcal{M} , and let $\mathcal{S} = \mathcal{D}^{\perp}$. Then there exists a set of maps \mathcal{X} such that $\mathcal{X}^{\perp} = \mathcal{S}^{\perp}$.*

PROOF. By abuse of notation, we also denote by \mathcal{S} the full subcategory of $\text{Arr}\mathcal{M}$ generated by the class \mathcal{S} . Since \mathcal{M} is locally presentable, $\text{Arr}\mathcal{M}$ is also locally presentable. Then, assuming Vopěnka's principle, it follows from [1, Theorem 6.6] that \mathcal{S} is bounded, i.e., it has a small dense subcategory. We have shown in Lemma 1.3 that there exists a regular cardinal λ such that \mathcal{S} is closed under λ -directed colimits in the category $\text{Arr}\mathcal{M}$. Hence, by [1, Corollary 6.18], the full subcategory generated by \mathcal{S} in $\text{Arr}\mathcal{M}$ is accessible. Thus, for a certain regular cardinal $\lambda_0 \geq \lambda$, the class \mathcal{S} contains a set \mathcal{X} of λ_0 -presentable objects such that every object of \mathcal{S} is a λ_0 -directed colimit of objects of \mathcal{X} .

Since $\mathcal{X} \subset \mathcal{S}$, we have $\mathcal{X}^{\perp} \supset \mathcal{S}^{\perp}$ and $(\mathcal{X}^{\perp})^{\perp} \subset (\mathcal{S}^{\perp})^{\perp} = \mathcal{S}$. Our aim now is to show the reverse inclusion $(\mathcal{X}^{\perp})^{\perp} \supset \mathcal{S}$. By Lemma 1.3, $(\mathcal{X}^{\perp})^{\perp}$ is closed under λ -directed colimits. Hence $(\mathcal{X}^{\perp})^{\perp}$ is also closed under λ_0 -directed colimits and every element of \mathcal{S} is a λ_0 -directed colimit of elements of \mathcal{X} . Then we can choose \mathcal{X} as our generating set. \square

THEOREM 1.5. *Let \mathcal{M} be a left proper, combinatorial, simplicial model category. If Vopěnka's principle is assumed true, then for any (possibly proper) class of maps \mathcal{S} there is a homotopy localization with respect to \mathcal{S} .*

PROOF. By Lemma 1.4, there exists a set \mathcal{X} of maps in \mathcal{M} such that $\mathcal{X}^{\perp} = \mathcal{S}^{\perp}$. Then the homotopy localization with respect to \mathcal{X} , which exists by Theorem 1.1, is an \mathcal{S} -localization. \square

Thus, the statement of Theorem 1.5 is a positive answer to the orthogonal subcategory problem in sufficiently good model categories.

2. Idempotent functors and simplicial orthogonality

The next theorem is motivated by results of Dror Farjoun in [12]. We consider a model category \mathcal{M} and assume, as in the beginning of the previous section, that a functorial choice of a homotopy function complex $\text{map}(X, Y)$ for all X and Y has been made.

In what follows, if $f: A \rightarrow B$ is a map and X is an object, we denote by $\text{map}(f, X)$ the map of simplicial sets $\text{map}(B, X) \rightarrow \text{map}(A, X)$ induced by f . If $\eta: F \rightarrow G$ is a natural transformation between two functors and H is another functor, then $\eta H: FH \rightarrow GH$ denotes the natural transformation given by $(\eta H)_X = \eta_{HX}$ for every object X , and $H\eta: HF \rightarrow HG$ denotes the natural transformation given by $(H\eta)_X = H\eta_X$ for all X .

THEOREM 2.1. *Let \mathcal{M} be any model category. Let L be an endofunctor in the homotopy category $\text{Ho}\mathcal{M}$ with the following properties:*

- (a) *There is a natural transformation $\eta: \text{Id} \rightarrow L$ in HoM such that $L\eta = \eta L$ and $L\eta: L \rightarrow LL$ is an isomorphism on all objects.*
- (b) *There is a map $l_{X,Y}: \text{map}(X, Y) \rightarrow \text{map}(LX, LY)$ for all X, Y , which is natural in both variables up to homotopy.*
- (c) *$\text{map}(\eta_X, LY) \circ l_{X,Y} \simeq \text{map}(X, \eta_Y)$ for all X and Y .*

Then the map

$$\text{map}(\eta_X, LY): \text{map}(LX, LY) \rightarrow \text{map}(X, LY)$$

is a weak equivalence for all X, Y .

PROOF. Let us write $Z = LY$ for simplicity. The assumption (a) says precisely that L is idempotent in the homotopy category HoM . Hence, among other consequences of this fact, $\eta_Z: Z \rightarrow LZ$ is an isomorphism in HoM . Then $\text{map}(A, \eta_Z)$ is a weak equivalence of fibrant simplicial sets for every A , hence a homotopy equivalence. Choose a homotopy inverse

$$\xi_{A,Z}: \text{map}(A, LZ) \rightarrow \text{map}(A, Z)$$

of $\text{map}(A, \eta_Z)$ for each A . We claim that $\xi_{LX,Z} \circ l_{X,Z}$ is now a homotopy inverse of $\text{map}(\eta_X, LY)$. The proof proceeds as in [6, Theorem 2.4]. On one hand, by the naturality of l ,

$$\xi_{LX,Z} \circ l_{X,Z} \circ \text{map}(\eta_X, Z) \simeq \xi_{LX,Z} \circ \text{map}(L\eta_X, LZ) \circ l_{LX,Z}.$$

Then, using the fact that $L\eta = \eta L$ in HoM and assumption (c), we obtain

$$\begin{aligned} \xi_{LX,Z} \circ \text{map}(L\eta_X, LZ) \circ l_{LX,Z} &\simeq \\ \xi_{LX,Z} \circ \text{map}(\eta_{LX}, LZ) \circ l_{LX,Z} &\simeq \xi_{LX,Z} \circ \text{map}(LX, \eta_Z) \simeq \text{id}. \end{aligned}$$

On the other hand,

$$\text{map}(\eta_X, Z) \circ \xi_{LX,Z} \circ l_{X,Z} \simeq \xi_{X,Z} \circ \text{map}(X, \eta_Z) \circ \text{map}(\eta_X, Z) \circ \xi_{LX,Z} \circ l_{X,Z}.$$

Since composition with η_X on the left and composition with η_Z on the right commute, we obtain

$$\begin{aligned} \xi_{X,Z} \circ \text{map}(X, \eta_Z) \circ \text{map}(\eta_X, Z) \circ \xi_{LX,Z} \circ l_{X,Z} &\simeq \\ \xi_{X,Z} \circ \text{map}(\eta_X, LZ) \circ \text{map}(LX, \eta_Z) \circ \xi_{LX,Z} \circ l_{X,Z} &\simeq \\ \xi_{X,Z} \circ \text{map}(\eta_X, LZ) \circ l_{X,Z} &\simeq \end{aligned}$$

Finally, using (c) again,

$$\xi_{X,Z} \circ \text{map}(\eta_X, LZ) \circ l_{X,Z} \simeq \xi_{X,Z} \circ \text{map}(X, \eta_Z) \simeq \text{id},$$

which completes the proof. \square

Assumptions (b) and (c) in Theorem 2.1 need not be satisfied by arbitrary idempotent functors in HoM , not even by those derived from functors in \mathcal{M} . Recall that a functor F in a simplicial model category is called *simplicial* or *continuous* if it is equipped with natural maps of simplicial sets

$$l_{X,Y}^F: \text{Map}(X, Y) \rightarrow \text{Map}(FX, FY)$$

preserving composition and identity; see [19, IX.1] or [20, 9.8]. A natural transformation $\zeta: F \rightarrow G$ of simplicial functors is called *simplicial* if

$$(4) \quad \text{map}(\zeta_X, GY) \circ l_{X,Y}^G = \text{map}(FX, \zeta_Y) \circ l_{X,Y}^F$$

for all X and Y ; cf. [19, IX.1].

Now, if \mathcal{M} is simplicial and $L: \mathcal{M} \rightarrow \mathcal{M}$ is a simplicial functor preserving weak equivalences and equipped with a simplicial natural transformation $\eta: \text{Id} \rightarrow L$ rendering L idempotent in $\text{Ho}\mathcal{M}$, then L and η fulfill the conditions of Theorem 2.1 in $\text{Ho}\mathcal{M}$. Thus, we may view conditions (b) and (c) in Theorem 2.1 as “continuity up to homotopy” of L and η , respectively. As we have shown, continuity up to homotopy is sufficient for the validity of Dror Farjoun’s result [12, Theorem 2.1]. In fact we have extended it to arbitrary simplicial model categories.

Now we use Proposition 6.4 in [23] to show that the assumptions (b) and (c) in Theorem 2.1 hold automatically in most cases of interest. Let \mathcal{M} be any model category and let $s\mathcal{M}$ denote the category of simplicial objects over \mathcal{M} . The *canonical model structure* on $s\mathcal{M}$ is the one where every level equivalence is a weak equivalence, the cofibrations are the Reedy cofibrations, and the fibrant objects are the homotopically constant Reedy fibrant objects (see [23] for motivation and further details). This model structure need not exist; however, when it exists, $s\mathcal{M}$ is a simplicial model category that is Quillen equivalent to \mathcal{M} . Moreover, the simplicial model category structure on $s\mathcal{M}$ is unique up to simplicial Quillen equivalence.

Sufficient conditions for the existence of the canonical model structure in $s\mathcal{M}$ were given in [23], and other sufficient conditions can be found in [14]. Pointed model categories where the suspension functor and the loop functor are inverse equivalences on the homotopy category are called *stable*. According to [23, Proposition 4.5], if \mathcal{M} is a proper, cofibrantly generated, stable model category, then the canonical model structure on $s\mathcal{M}$ exists. Likewise, as shown in [14], if \mathcal{M} is left proper and combinatorial, or left proper and cellular, then the canonical model structure on $s\mathcal{M}$ exists.

THEOREM 2.2. *Let \mathcal{M} be a cofibrantly generated simplicial model category where the canonical model category structure exists in $s\mathcal{M}$. Let L be an endofunctor in \mathcal{M} equipped with a natural transformation $\eta: \text{Id} \rightarrow L$. Suppose that L preserves weak equivalences, takes fibrant values, and η renders it idempotent in the homotopy category $\text{Ho}\mathcal{M}$. Then L is a homotopy localization with respect to the class of maps η_X for all X .*

PROOF. Let us denote by L' the simplicial approximation of L given by [23, Corollary 6.5]. Thus, $L'X = |Q\hat{L}\text{Sing } RX|$ for each object X , where the notation is as follows. The singular functor Sing is defined as $(\text{Sing } X)_n = X^{\Delta[n]}$ for all n ; the realization functor $|-|$ is its left adjoint; \hat{L} is the prolongation of L over $s\mathcal{M}$; R is a fibrant replacement functor in \mathcal{M} , and Q is a simplicial cofibrant replacement functor in $s\mathcal{M}$. By its construction, L' is a simplicial functor, since it is a composite of simplicial functors (see [23] for details), and there is a ziz-zag of weak equivalences between LX and $L'X$ for all X .

Although it is not explicitly stated in [23], if $\zeta: F \rightarrow G$ is any natural transformation of functors that preserve weak equivalences, then the above construction yields a natural transformation $\zeta': F' \rightarrow G'$ which is itself simplicial. Thus, in our case, there is a simplicial natural transformation $\eta': \text{Id}' \rightarrow L'$ (where Id' need not be the identity). Therefore, although L' need not be a coaugmented functor in \mathcal{M} , it follows that L and η fulfill the conditions of Theorem 2.1 in the homotopy category $\text{Ho}\mathcal{M}$, since L and L' define isomorphic functors in $\text{Ho}\mathcal{M}$. The conclusion of Theorem 2.1 implies then that L is a homotopy localization with respect to the class of maps of the form η_X for all X . \square

Now the results of the previous section yield the following answer to Dror Farjoun's problem in sufficiently good model categories.

THEOREM 2.3. *Assuming Vopěnka's principle, any homotopy localization in a left proper, combinatorial, simplicial model category is an \mathcal{X} -localization for some set of maps \mathcal{X} .*

PROOF. Under these assumptions, the canonical model structure exists in $s\mathcal{M}$ by [14]; cf. [23, Remark 3.8]. Therefore, Theorem 2.2 can be used and Lemma 1.4 completes the argument. \square

This result applies to a useful case not previously established in the literature, namely to the stable homotopy category of Adams–Boardman, by using, for example, the model category of symmetric spectra based on simplicial sets.

In the model categories of simplicial sets or spectra, the set \mathcal{X} of maps given by Theorem 2.3 can be replaced by a single map f , namely the coproduct $\coprod_{g \in \mathcal{X}} g$ of all maps in \mathcal{X} . In a general model category, one has to be more careful, in view of the next counterexample.

Consider the model category which is a product of two copies of the category of simplicial sets, i.e., the category of diagrams of simplicial sets over the discrete category with two objects, equipped with the Bousfield–Kan model structure (where fibrations and weak equivalences are objectwise). Take $S = \{f, g\}$ for

$$f: (\emptyset, \emptyset) \longrightarrow (*, \emptyset) \quad \text{and} \quad g: (\emptyset, *) \longrightarrow (\emptyset, * \coprod *).$$

An object (X, Y) is S -local if and only if X and Y are fibrant, X is contractible and Y is either contractible or empty.

Suppose that there exists a map

$$h: (A, B) \longrightarrow (C, D)$$

such that any S -local object is also h -local, and vice versa. The object (X, \emptyset) is h -local if and only if X is contractible. This condition implies that both B and D are empty; otherwise, for any simplicial set Z , either contractible or not, the object (Z, \emptyset) would be h -local. But in this case any object (X, Y) with contractible X becomes h -local, hence the contradiction. Note however that, in order to ensure that every set of maps yields the same localization as their coproduct, it is enough to assume that the set of maps $X \rightarrow Y$ is nonempty for all X and Y in the model category under consideration.

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